A note on the Sorgenfrey line

Ivan D. Arandjelović

Abstract

In this paper, by using Cantor’s principle of nested intervals, we give a new and simple proof that the Sorgenfrey line is a topological space of the second Baire category. One application of this result in asymptotic analysis is also given.

The Sorgenfrey line \((S_1)\) is the real line with the topology whose base is the family of intervals
\[ [a, b), \quad -\infty < a < b < \infty; \ a, b \in \mathbb{R}. \]
This space was introduced by R.H. Sorgenfrey [4] as an example of a normal topological space whose square is not normal. It is a non-metrizable, totally disconnected, paracompact, normal and Lindelöf topological space. The Sorgenfrey plane \(S_2 = S_1 \times S_1\) is not a paracompact, normal and Lindelöf space.

A topological space \(X\) is of the first Baire category if it is a countable union of nowhere dense subsets of \(X\); otherwise, \(X\) is of the second Baire category. It follows from the definition that a topological space \(X\) is of the second Baire category, or shorter Baire’s space, if and only if the intersection of any countable family of open dense sets in \(X\) is dense in \(X\). Classical examples of Baire’s spaces are complete metric spaces and locally compact topological spaces. In this note we give a new proof that the Sorgenfrey line is a Baire space.

**Proposition 1** The Sorgenfrey line is a Baire space.

**Proof.** Let \((A_n : n \in \mathbb{N})\) be a sequence of sets such that each of them is open and dense in the Sorgenfrey line and \(A = \cap_{n \in \mathbb{N}} A_n\). Let \(a, b \ (\infty < a < b < \infty)\) be arbitrary real numbers. We shall prove that the interval \([a, b)\) contains a point of \(A\). The set \(A_1^n\) is nowhere dense in \(S_1\) and so there exist real numbers \(a_1, b_1 \in S_1\) such that
\[ a < \frac{2a + b}{3} < a_1 < b_1 < \frac{a + 2b}{3} < b \quad \text{and} \quad (a_1, b_1) \subseteq A_1. \]

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Continuing this procedure we obtain sequences \( \{a_k\}, \{b_k\} \) such that \( [a_k, b_k) \subseteq A_k \) and
\[
a_k < \frac{2a_k + b_k}{3} < a_{k+1} < b_{k+1} < \frac{a_k + 2b_k}{3} < b_k.
\]
From the Cantor principle of nested intervals it follows that there exists only one real number \( \lambda \) such that
\[
\lambda \in \bigcap_{k \in \mathbb{N}} [a_k, b_k].
\]
So we have that
\[
A \cap [a, b) \neq \emptyset
\]
for any interval \([a, b)\) which implies that \( A \) is dense in \( S_1 \).

From this proposition it follows that the Sorgenfrey plane \( S_2 \) is a Baire space.

A function \( f : X \to \mathbb{R} \) from a topological space \( X \) into the real line is lower semicontinuous at a point \( x_0 \in X \) if and only if
\[
\liminf_{x \to x_0} f(x) \geq f(x_0).
\]
\( f \) is lower semicontinuous on a set \( A \subseteq X \) if it has this property at each point of \( A \). \( f : X \to \mathbb{R} \) is a lower semicontinuous function on \( X \) if and only if the set \( \{x \in X : f(x) < r\} \) (\( \{x \in X : f(x) > r\} \)) is open for each \( r \in \mathbb{R} \). The least upper bound of a family of continuous functions on a Baire space is lower semicontinuous, and the set of points in which it is bounded is open and dense in this space ([1], [3]).

The next statement is an extension of Theorem 2 from the paper [2].

**Corollary 1** Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of positive real numbers such that
\[
\limsup_{m,n \to \infty} \frac{a_{\lfloor \alpha m \rfloor}}{b_{\lfloor \beta n \rfloor}}
\]
is finite for each \( \alpha, \beta > 0 \), and let \( I_1, I_2 \) be compact intervals in \((0, \infty)\). Then
\[
\limsup_{m,n \to \infty} \sup_{\alpha \in I_1, \beta \in I_2} \frac{a_{\lfloor \alpha m \rfloor}}{b_{\lfloor \beta n \rfloor}} < \infty.
\]

**Proof.** The function \( R_{m,n} : [0, \infty) \to [0, \infty) \) defined by:
\[
R_{m,n}(\alpha, \beta) = \frac{a_{\lfloor \alpha m \rfloor}}{b_{\lfloor \beta n \rfloor}}
\]
is continuous on \([0, \infty)^2\) in the \(S_2\) topology for any \(m, n = 1, 2, \ldots\). The family of functions \(\{R_{m,n}\}\) is bounded for each \(\alpha, \beta \in [0, \infty)\) which implies that the function \(r : [0, \infty)^2 \to [0, \infty)\) defined by

\[
   r(\alpha, \beta) = \limsup_{m,n \to \infty} \frac{a_{[\alpha m]}}{b_{[\beta n]}}
\]

is lower semicontinuous on \([0, \infty)^2\) in the \(S_2\) topology.

So for any compact interval \(I\) with \(I^2 \subseteq [0, \infty)^2\) there exists an open and dense subset \(I' \subseteq I^2\) such that:

\[
   \sup_{(\alpha, \beta) \in I'} r(\alpha, \beta) < \infty.
\]

If \((\alpha, \beta) \in I'\), there exists a sequence \(\{(\alpha_m, \beta_n)\} \subseteq I'\) such that \((\alpha_m, \beta_n) \to (\alpha, \beta)\) and \([\alpha_m, m] = [\alpha m], \ [\beta_n, n] = [\beta n]\). This implies that \((\alpha, \beta) \in I'\). So \(I'\) is a closed set which implies that \(I^2 = I'\).

\[\text{References}\]


Faculty of Mechanical Engineering
27. marta 80
11000 Beograd
Yugoslavia