On the acoustics of tuning forks

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Tuning forks can vibrate in many different modes in which the tines move either in the plane or perpendicular to the plane of the fork. Symmetrical modes can be modeled by the motion of two cantilever beams, antisymmetrical modes by the motion of a beam with free ends. A tuning fork vibrating in its fundamental mode is approximately a linear quadrupole sound source whose strength can be increased by use of a baffle or by touching the stem to a soundboard. The motion of the stem includes strong components at both the fundamental frequency and its second harmonic. Slight alterations in a tuning fork can enhance or suppress either of these components. At large amplitudes, the tines vibrate nonsinusoidally, the nth harmonic increasing approximately as the nth power of the fundamental.

I. INTRODUCTION

Tuning forks are familiar to musicians and physicists, in fact to anyone who deals with sound and hearing. For many years, they have enjoyed popularity as frequency standards because they are stable, inexpensive, portable, and require no electrical power. A simple description of tuning fork vibrations appears in many general physics texts, but more advanced mechanics books seldom mention the more subtle features of their mechanical or acoustical behavior.

In this paper we will attempt to briefly review the mechanical and acoustical behavior of tuning forks, especially the normal modes of vibration and the nonlinear transfer of energy from the tines to the stem. The effects of attaching small masses and bending the tines will be discussed, and simple laboratory and demonstration experiments described.

II. MODES OF VIBRATION

The aluminum tuning forks most generally used in our general physics laboratories have two rectangular 9×7-mm tines spaced 10 mm apart. The length of the tines is determined by the frequency of the fork. Attached to the base where the tines join is a cylindrical stem approximately 8 mm in diameter and 45 mm long. To a first approximation, the fork can be considered to be made up of two cantilevered bars joined together at the base.

For convenience, we classify the modes of vibration of a tuning fork into four groups: (1) symmetrical modes in the plane of the fork; (2) antisymmetrical modes in the plane; (3) symmetrical modes out of the plane; (4) antisymmetrical modes out of the plane. The first three in-plane modes are shown in Fig. 1. Modes (a) and (c) belong to the first group (symmetrical), while mode (b) belongs to the second group above. Mode (a) is the principal mode, while mode (c) is often referred to as the "clang" mode.[1] Mode (b) would be the predominant one if a "sound post" were inserted between the tines, forcing them to move in the same direction.

In the symmetrical in-plane modes [e.g., (a) and (c) in Fig. 1], the motion of each tine more or less resembles that of a cantilever beam whose modal frequencies f_n (for a thin beam) are given by

$$f_n = (\pi K/8L^2)\sqrt{E/\rho}\,[1.194^2, 2.988^2, 5^2, 7^2, \ldots, (2n-1)^2], \quad (1)$$

where E is Young's elastic modulus, ρ is density, L is the length of the tines, and K is the radius of gyration of the beam cross section ($1/\sqrt{12}$ times the thickness for a bar with rectangular cross section). The stem has a vertical component of motion at twice the frequency of the tines, although the amplitude of this second-order motion is small except in the principal mode [Fig. 1(a)].

In the antisymmetrical in-plane modes [such as in Fig. 1(b)], the entire fork bends in the manner of a beam with free ends whose modal frequencies f_n (for a thin beam of uniform cross section) are given by

$$f_n = (\pi K/8L^2)\sqrt{E/\rho}\,[3.011^2, 5^2, 7^2, \ldots, (2n+1)^2]. \quad (2)$$

Of course the fork does not have a uniform cross section along its length (which now includes the stem as well as the tines), so the nodes will not occur at equal distances from the two ends as they do in a uniform beam. In the antisymmetrical in-plane modes, the stem moves from side to side, as shown in Fig. 1(b).

The symmetrical and antisymmetrical out-of-plane modes would be expected to behave more or less the same as the in-plane modes, although the bending stiffness is

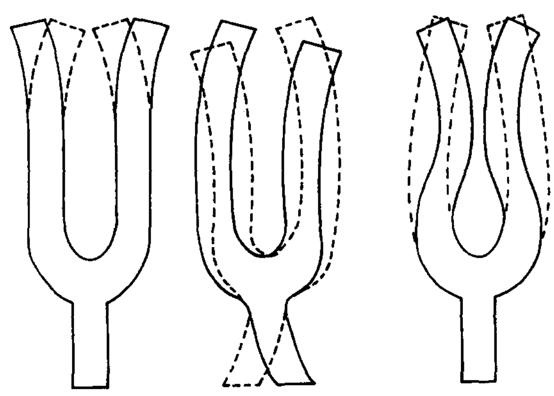

Fig. 1. First three in-plane vibrational modes of a tuning fork. (a) is the principal mode; (c) is the "clang" mode.

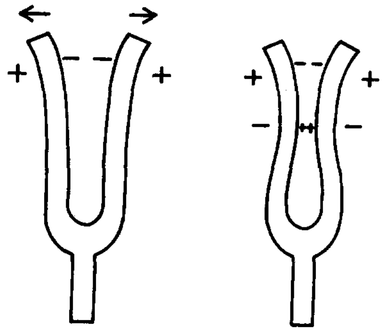

Fig. 2. (a) Tuning fork vibrating in its principal mode acts as a linear quadrupole source. (b) Tuning fork vibrating in its "clang" mode acts as two linear quadrupoles, giving it an octupole character.

different, and so the frequencies are different from the corresponding families of in-plane modes.

The relative amplitudes of excitation of the various modes depend upon where the tuning fork is struck. Striking near a node provides the least excitation for a given mode, while striking near an antinode generally excites the mode strongly.

III. SOUND RADIATION

In its principal mode [Fig. 1(a)], a tuning fork essentially acts as a linear quadrupole. As the tines move outward, the air pressure between them decreases, while the pressure on the outer surfaces increases, as shown in Fig. 2(a). This can be demonstrated by passing the fork through a slit in a sheet of paper, as shown in Fig. 3. When either tine is centered in the slit, sound output audibly increases, since a more efficient dipole source is created. The clang mode acts as two unequal linear quadrupoles, as shown in Fig. 2(b), giving it an octupole, as well as a quadrupole character.

Radiation from a quadrupole source is quite inefficient, and thus the direct sound radiation from a tuning fork is weak, especially one of low frequency. Quadrupole radiation efficiency is proportional to the 6th power of frequency, however, so the clang mode (which has a frequency roughly 6 times greater than that of the principal mode) radiates roughly 50 000 ($\approx 6^6$) times more efficiently than the principal mode. Thus even though the amplitude of the clang mode is small, it is easily heard when the fork is first struck. To hear the principal note of a tuning fork clearly, the stem is usually placed in contact with some type of sounding board.

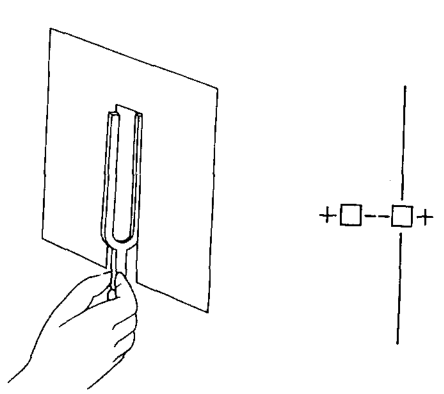

Fig. 3. When either tine of a tuning fork is centered in the slot, the paper acts as a "baffle," and the tuning fork radiates more like a dipole source (or two monopole sources if the paper is large enough).

IV. MOTION OF THE STEM

The stem of a tuning fork moves in a rather complex way when the tines of the fork vibrate. In the symmetric modes, the motion of the stem is mainly longitudinal [Fig. 1(a)]; in the antisymmetric modes, its motion is mainly transverse [Fig. 1(b)].

The longitudinal motion of the stem is not sinusoidal. There is a strong second harmonic component in its motion, and higher harmonics appear as well. The mechanism mainly responsible for generating the second harmonic is illustrated by the simple mechanical model in Fig. 4. Imagine the two tines of the fork to be bars swinging back and forth. Each time they make one full trip back and forth, the stem moves up and down twice. The amplitude of the stem is much smaller than that of the tines, but it increases with the square of the tine amplitude (just as the second harmonic of the tine itself), so at large amplitude it makes its presence known.

The relative amplitudes of the harmonics in the stem motion are strongly dependent on the amplitude of vibration. For a striking blow of normal strength, the fundamental and second harmonic will generally have about the same amplitude, but after a hard blow the second harmonic can be considerably larger.

When a tuning fork is altered, so as to change its symmetry, the balance between the fundamental and second harmonic in the stem motion changes. We found that bending the tines inward, for example, inhibits the second harmonic. The fork now tends to oscillate between frame 4 and frame 5 in Fig. 4, driving the stem up and down at the fundamental frequency. Adding small masses to the inner surfaces of the tines, on the other hand, inhibits the fundamental, as noted by Rayleigh.[2] Adding mass to the outer surface of the tines has less effect.

When a piano tuner touches the stem of a tuning fork against a piano soundboard, it is the longitudinal motion of the stem that drives the soundboard. If the fork is given a hard blow [Fig. 5(b)], the second-harmonic component of the stem motion may be greater than the fundamental. However, it dies out faster, and so after a certain time the fundamental dominates, as in Fig. 5(a). For a soft blow, the motion of the tine and stem is more like (a) from the very beginning. The "clang mode" also drives the stem at its second harmonic frequency, but the amplitude is generally very small, so it is not a factor in the sound radiated by the soundboard.

V. NONLINEAR BEHAVIOR

At sufficiently large amplitudes, most oscillators exhibit nonlinear behavior. Two common nonlinear phenomena are: a transition from sinusoidal to nonsinusoidal motion (leading to generation of harmonics) and a shift in modal frequency. Both of these effects can be observed in tuning forks at moderately large amplitudes.

Modal frequencies in oscillators can shift either upward or downward at large amplitude, depending upon whether the stiffness increases or decreases with increasing amplitude. If the effective spring constant increases with amplitude (hardening spring system), the frequency rises; if the

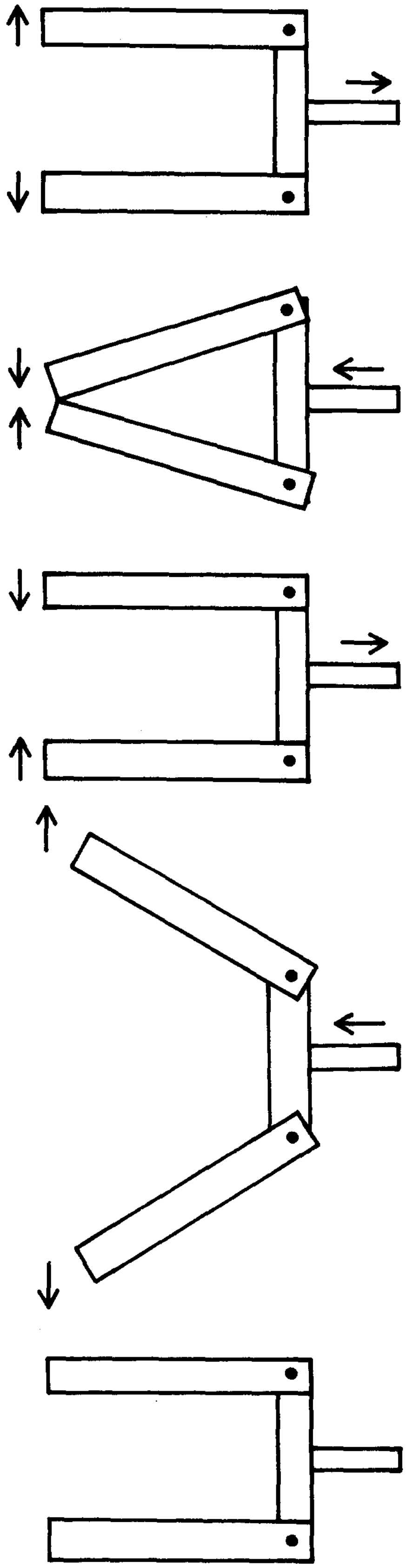

Fig. 4. Mechanical model of a tuning fork illustrating how motion of the tines at a frequency f causes the stem to move at a frequency $2f$.

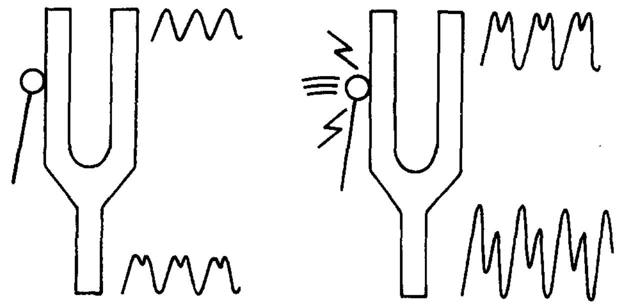

Fig. 5. (a) A soft blow generates a small second harmonic in the stem motion; (b) a hard blow generates a small second harmonic in the tine motion and a large second harmonic in the stem motion.

spring constant decreases with amplitude (softening spring system), the frequency falls. A vibrating string generally behaves as a hardening spring system, while a pendulum behaves as a softening spring system. An interesting example from the musical world is a pair of gongs used in Chinese opera. One gong shows hardening spring behavior, while the other shows softening spring behavior. Thus the pitch of one gong glides upward after striking, while the pitch of the other one glides downward.[3] A discussion of nonlinear vibrating systems is given in Ref. 4 Chap. 5.

In a cantilever beam the first mode of vibration shows hardening spring behavior, while the second mode shows softening spring behavior.[5] In a free–free beam, on the other hand, both modes show hardening spring behavior. This suggests that some modes of a tuning fork may be higher than normal immediately after striking a hard blow while others may be lower.

VI. EXPERIMENTAL STUDIES

A. Modes of vibration

A small (0.22 g) NdFeB magnet was attached to one tine of the fork, and positioned in the magnetic field of a small coil with 300 turns. A sinusoidal current was supplied to the coil by an audio amplifier, and the near-field sound was scanned with a small electret microphone to locate the nodes and antinodes. The fork was freely suspended by rubber bands. The magnet was mounted in two different orientations to excite in-plane and out-of-plane modes. The experimental setup is similar to that described in Ref. 6.

The modal frequencies of the in-plane modes of a typical 384-Hz tuning fork are shown in Fig. 6. The symmetrical in-plane modes are plotted against $2n - 1$ in accordance with Eq. (1). Note that the first mode lies at 1.194 rather than 1, as predicted for a cantilevered beam. The antisymmetrical modes are plotted against $2n + 1$ in accordance with Eq. (2). Note that each series of mode frequencies can be reasonably well fit to a line with a slope of 2. The symmetrical modes are higher in frequency, because the effective length in Eq. (1) is essentially the length of the tines, whereas the effective length of the antisymmetrical modes in Eq. (2) is the entire length of the fork.

The modal frequencies and locations of the nodes for the same fork are given in Table I. The mass of the small magnet has lowered all the modal frequencies slightly (e.g., the fundamental from 384 to 383 Hz). The tines of this fork are

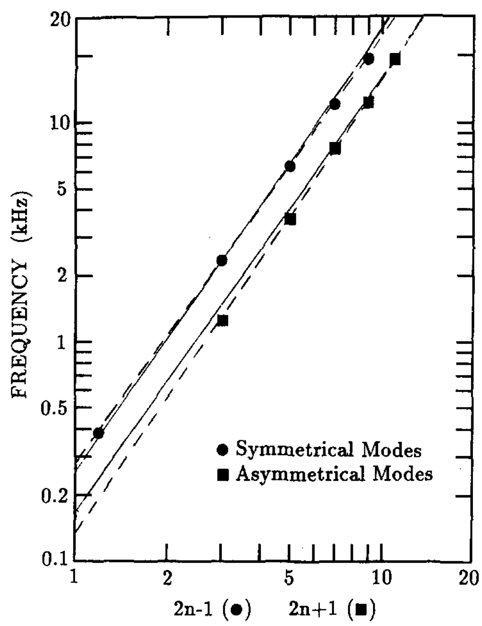

Fig. 6. Modal frequencies of in-plane vibrational modes of a 384-Hz tuning fork. Symmetrical modes are plotted vs $2n-1$ while antisymmetric modes are plotted vs $2n+1$. Solid lines have a slope of 2.

122 mm long, the stem is 43 mm long, and the total length of the fork is 183 mm.

The most significant frequency ratio is the ratio of the "clang" mode to the fundamental mode. This ratio is relatively constant in forks of different lengths from the same manufacturer. In the 384-Hz fork of Table I, this ratio is 6.13; in a 320-Hz fork (189 mm long) it is 6.16; in a 256-Hz fork (212 mm long), it is 6.20. This ratio, of course, depends upon the design of the tuning fork. (It is reported that some manufacturers design the base of the fork so that these frequencies have the harmonic ratio 6.00; we have not investigated this.)

The out-of-plane mode frequencies are shown in Fig. 7. They deviate considerably more from the lines drawn with a slope of 2 than the in-plane modes in Fig. 6 do.

B. Motion of the stem

The motion of the stem was measured by attaching a small accelerometer (PCB 309A, mass = 1 g) to the bottom of the stem. Figure 8 shows the amplitudes of the fundamental and second harmonic of a 320-Hz tuning fork for blow strengths covering a rather wide range. Note that the second harmonic amplitude is proportional to the square of the fundamental amplitude (i.e., the line in Fig. 8 has a slope $p=2$), and that their amplitudes become equal slightly above the range shown.

In a related experiment, the accelerometer was attached near the center of a 50-cm square sheet of 1/4-in. plywood and the stem of the 320-Hz fork was touched to the soundboard about 5 cm away. Figure 9 shows the amplitudes of the fundamental and second harmonic. Again the second harmonic amplitude is proportional to the square of the fundamental amplitude, but in this case it exceeds the fundamental over most of the range of blow strength. In other words, the second harmonic component of motion is transferred more efficiently than the fundamental to the wood sounding board.

C. Nonlinear motion of the tines

In this experiment, a 1-g accelerometer was attached to one tine of a 320-Hz tuning fork and a 1-g mass to the other tine. A second accelerometer was attached to the stem. Figure 10 shows the amplitudes of the 2nd, 3rd, and 4th harmonics of time motion compared to the fundamental amplitude. The 2nd, 3rd, and 4th harmonics are seen to be nearly proportional to the 2nd, 3rd, and 4th powers of the fundamental (i.e., their amplitudes lie along lines with slopes of 2, 3, and 4), as predicted by simple nonlinear vibration theory.

The amplitudes of the stem and tines are compared in Fig. 11. The amplitude of the fundamental and 2nd harmonic motion of the stem are proportional to the first and second powers of the tine fundamental. The accelerometer attached to the stem has approximately twice the sensitivity of the one attached to the tine, so the stem fundamental and second harmonic amplitudes $A_1(s)$ and $A_2(s)$ can be expressed as

$$A_1(s) = 3.3\times10^{-3}A_1(t) \text{ and } A_2(s) = 7.8\times10^{-4}A_1^2(t),$$

where $A_1(t)$ is the amplitude of the fundamental component in the tine motion (in mm).

D. Altered tuning forks

Figure 12 compares the spectrum of the stem motion of a normal 320-Hz tuning fork with that of the same fork with

Table I. In-plane vibration modes of a tuning fork.

Mode number (n)	Type	Frequency (Hz)	Ratio	Node locations (mm from tip)					
1	symmetric	383	1.0	-					
2	symmetric	2346	6.1	29					
3	symmetric	6391	16.7	17	63				
4	symmetric	12075	31.5	15	45	83			
5	symmetric	19330	50.5	11	36	71	92		
1	antisymetric	1250	3.3	41	136				
2	antisymetric	3650	9.5	35	92	136			
3	antisymetric	7650	20.0	16	58	106	140		
4	antisymetric	12309	32.1	13	45	82	122	171	
5	antisymetric	19225	50.2	11	36	71	92	121	178

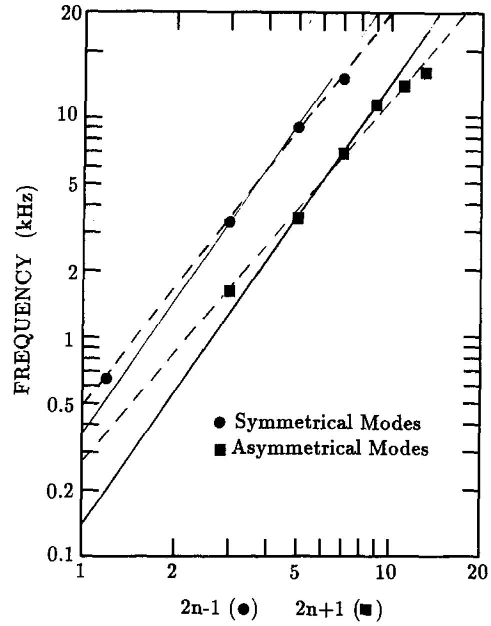

Fig. 7. Modal frequencies of out-of-plane vibrational modes of a 384-Hz tuning fork. Symmetrical modes are plotted vs $2n - 1$, antisymmetrical modes vs $2n + 1$. Solid lines have a slope of 2, while dashed lines are adjusted for the best fit.

the tines bent inward (3 mm spacing between tines at the tip compared to 10 mm at the base). Note that in the normal fork, the 2nd harmonic exceeds the fundamental by 5 dB; after bending the tines the 2nd harmonic amplitude is 15 dB below the fundamental for a blow of about the same strength.

Attaching 1-g masses to the inside of each tine near the tip of the fork lowered the fundamental amplitude of the stem motion by about 10 dB but left the amplitude of the second harmonic about the same. Attaching the same masses to the outside of each tine had little effect on either

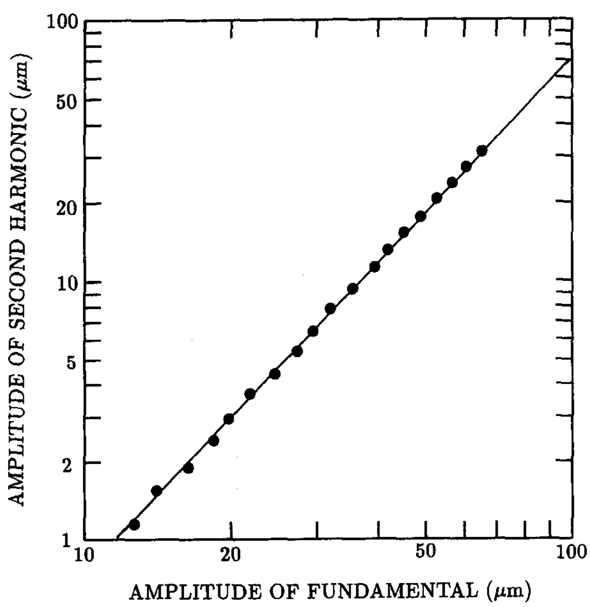

Fig. 8. Fundamental and second harmonic amplitudes of the stem of a 320-Hz tuning fork excited by blows of different strengths.

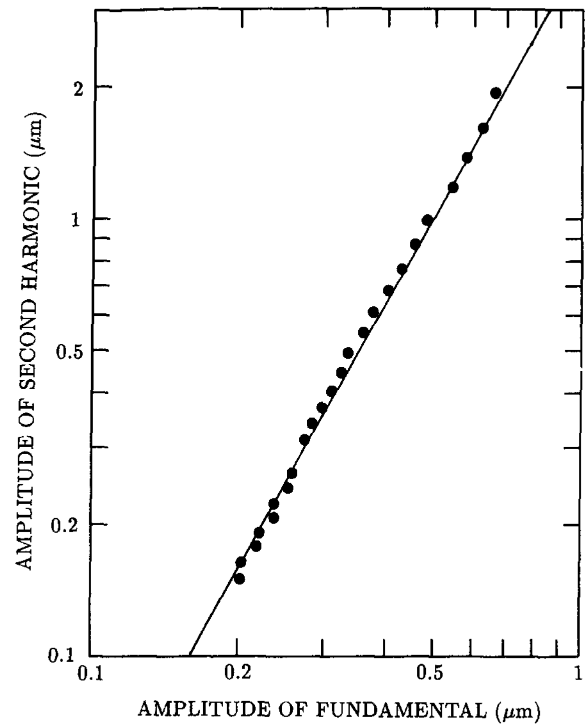

Fig. 9. Fundamental and second harmonic amplitudes measured by an accelerometer attached to a soundboard 5 cm from the stem of a 320-Hz tuning fork.

the fundamental or second harmonic (the fundamental was raised slightly, the second harmonic remained the same).

E. Resonators tuned to the fundamental and second harmonic

A popular classroom demonstration makes use of box resonators with one open end having a tuning fork attached

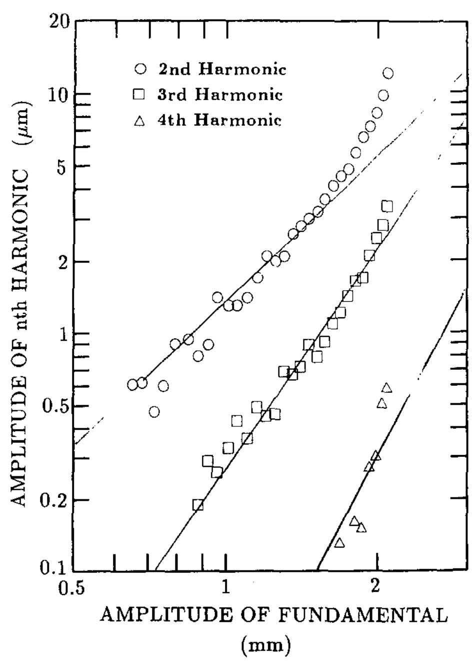

Fig. 10. Amplitudes of 2nd, 3rd, and 4th harmonics in the motion of the tines of a 320-Hz tuning fork compared to the fundamental amplitude.

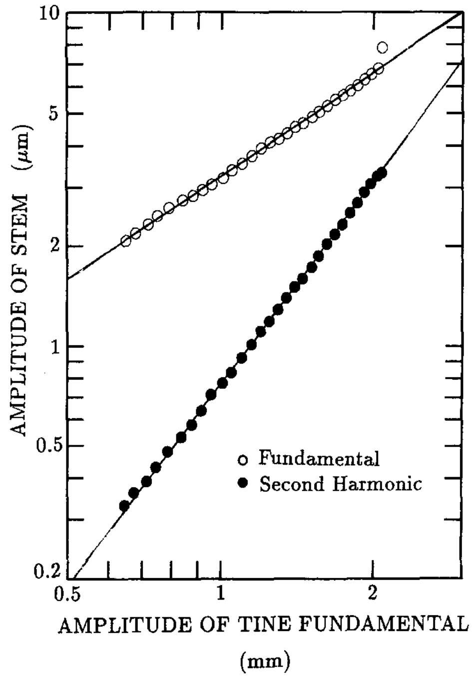

Fig. 11. Amplitudes of fundamental and second harmonic components in the stem motion in a 320-Hz tuning fork compared to the fundamental amplitude of the tines.

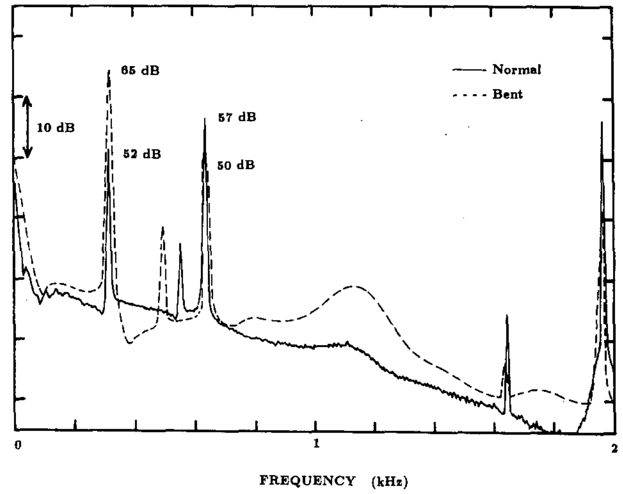

Fig. 12. Acceleration levels of the stem of a 320-Hz tuning fork compared to the same fork with the tines bent inward. Bending the tines increases the fundamental and decreases the second harmonic component of the motion.

to the upper surface. We have added a sliding plunger to such a box so that it can be tuned to either the fundamental or the second harmonic of the fork. Figure 13 shows sound spectra that result from a normal 320-Hz fork, one with the tines bent inward, and one with 1-g masses added to the inside surface of each tine.

VII. DISCUSSION OF EXPERIMENTAL RESULTS

It is interesting to note how well the in-plane modes in Fig. 6 follow the simple mechanical models (two cantilevered beams in the symmetric case and a single beam with free ends in the antisymmetric case) insofar as having frequencies proportional to $(2n-1)^2$ and $(2n+1)^2$, respectively. The node locations are also reasonably consistent with these simple models, considering the nonuniform cross section of the fork.

The out-of-plane modes in Fig. 7 conform less precisely to these models. The observed modal frequencies are proportional to $(2n-1)^{1.8}$ and $(2n+1)^{1.6}$, respectively. The reason for this difference is not clear. The antisymmetric out-of-plane modes occur at slightly higher frequencies than the corresponding in-plane modes, reflecting the greater bending stiffness for in-plane bending at the base of the fork.

The second harmonic component in the stem motion is very nearly proportional to the square of the fundamental component (and to the amplitude of the tines), which is consistent with the simple mechanical model for second harmonic generation in Fig. 4. It is interesting to note, by

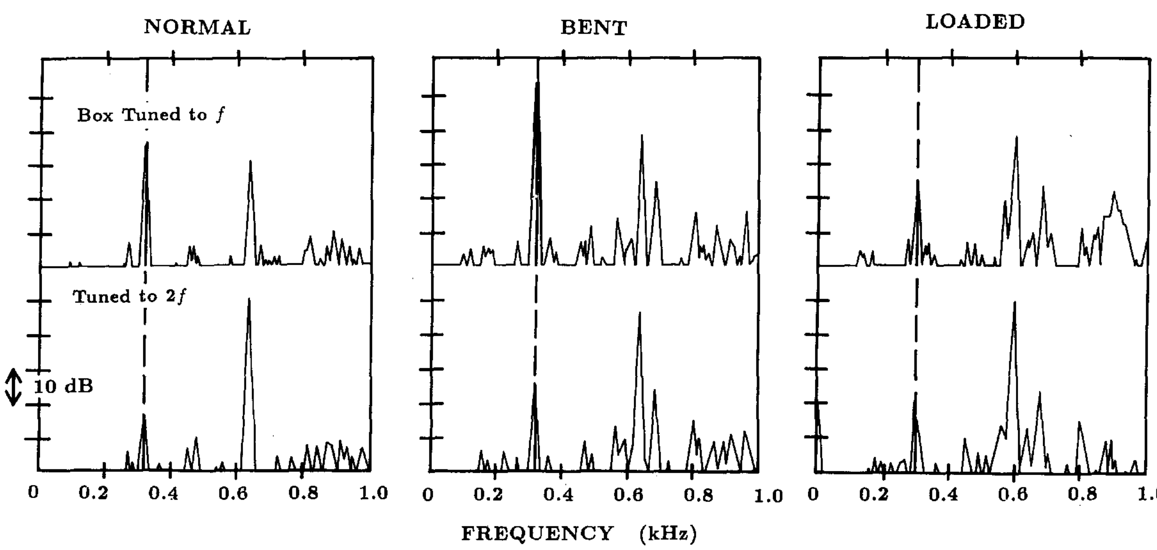

Fig. 13. Sound spectra radiated by a 320-Hz tuning fork with the stem in contact with an open-end box resonator that can be tuned to the fundamental or the second harmonic.

comparing Figs. 8 and 9, how much more efficiently energy is transferred to the soundboard at the second harmonic frequency than at the fundamental frequency.

The motion of the tines at large amplitudes becomes nonsinusoidal in a predictable way. The amplitudes of the 2nd, 3rd, and 4th harmonics are observed to be proportional to the 2nd, 3rd, and 4th powers of the fundamental amplitude. Bending the tines of a tuning fork inward was found to increase the fundamental component in the stem motion and to decrease the second harmonic. This result is consistent with the simple mechanical model shown in Fig. 4. However, it appears to be contrary to Rayleigh's observation that the fundamental component is decreased by bending the tines inward.[2]

Attaching small masses to the inner surface of the tines was found to decrease the fundamental component in the stem motion, as observed by Rayleigh.[2] Why our forks behaved in the same way as Rayleigh's in the mass-loading experiment but different from his when the tines were bent inward is not clear. Undoubtedly the behavior depends upon the exact shape of the fork, and further investigation using forks of different shapes would be worthwhile.

The difference between the effects of adding small masses to the inner and the outer surfaces of the tines needs further study. Apparently the difference is related to the direction in shift of the center of mass of the tines. Shifting it outward has much less effect than shifting it inward.

Since the observed tuning fork mode shapes resemble those of uniform beams with clamped–free (symmetrical modes) and free–free (antisymmetrical modes) end conditions, it is interesting to compare the observed mode frequencies with those calculated for such beams. Applying Eq. (1) to a clamped–free beam 122 mm long and 7 mm thick and using $\sqrt{E/\rho} = 5150$ m/s, gives mode frequencies of 391, 2451, 6864 Hz, etc. which compare quite favorably (within 7%) to the observed symmetric mode frequencies (see Table I).

The antisymmetric modes are more difficult to model, because the cross section of the fork is substantially different for the tines and the stem. For a uniform beam having a length (183 mm) equal to the total fork length and a thickness of 7 mm, application of Eq. (2) gives mode frequencies of 1106, 3051, 5979 Hz, etc., which are 12%–22% greater than the observed frequencies of the antisymmetric modes. This is to be expected, since the cross-sectional area of the stem is less than half the combined cross section of the two tines. This nonuniform cross section appears to affect the mode frequencies more than the mode shapes.

Simple apparatus for demonstrating the effect shown in Fig. 13 was displayed in an exhibit and competition of low-cost (under $25) apparatus held at the 1990 summer meeting of the American Association of Physics Teachers in Minneapolis.

VIII. CONCLUDING REMARKS

Tuning forks are very interesting mechanical oscillators. Although widely used by musicians, scientists, engineers, and medical practitioners, certain features of their mechanical behavior are not well understood. Hopefully this paper will add, in a modest way, to that understanding.

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THE PYTHAGOREAN THEOREM

In one of my trigonometry classes [at Cornell in the early 1940s] I discovered to my great surprise that a few students claimed never to have heard of the Pythagorean theorem. It may have been because of the quaint way in which I pronounced or mispronounced Pythagoras but this obvious explanation did not occur to me. "Did you know," I addressed the students, "that Pythagoras was so elated when he proved his theorem that he sacrificed a hundred oxen to the Greek gods in gratitude for the inspiration? And that since those days all oxen tremble when a truth is found? I was quoting a famous line of Heine's. Nobody smiled and from that moment on I relied on more elementary humor.

Mark Kac, *Enigmas of Chance—An Autobiography* (University of California, Berkeley, 1985), pp. 98–99.